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ISOLATION OF SINGULARITIES IN THE SOLUTION OF TWO-DIMENSIONAL PROBLEMS OF THE THEORY OF ELASTICITY IN IRREGULAR MULTIPLY CONNECTED DOMAINS*

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A general method is considered for isolating the singularities of the solutions of a plane problem of the theory of elasticity, a problem of the bending of thin elastic plates, and harmonic problems of the theory of elasticity in multiply connected domains with boundary breaks. The procedure is used to solve the problems by the method of compensating loads (MCL) or the method of integral equations of the first kind /1-6/.

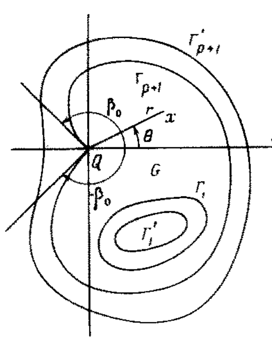

Fig.1

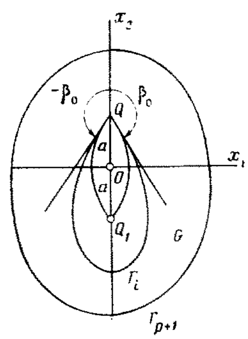

Fig.2

In the MCL the components of the directionally deformed state (DDS) are sought as potentials which are distributed along contours, spaced a certain distance from the domain boundary, rather than distributed along the boundary itself. When the potentials are substituted into the boundary conditions, systems of integral equations of the 1st kind are obtained in the unknown densities. Methods of regularizing the solution of these equations were considered in /3-6/. When the components of the DDS have singularities at corner points of the boundary, the modification of the MCL consists in adding to the potentials of the singular solutions of homogeneous boundary value problems for the auxiliary wedge-shaped domains /7-12/ (Fig.1). However, if the initial domain is multiply connected and the corner points are located on "interior" pieces of the boundary (Fig.2), the solutions for the wedge cannot be used in MCL. In this case the cut needed to isolate the one-valued branches of the

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functions necessarily intersects the domain. The components of DDS prove to be discontinuous inside the domain and on the "outer" part of the boundary, since, on circuiting a singular point along a closed contour, the singular solutions for the wedge return to different values at the start of the circuit even in the case of real eigenvalues.

The construction of singular solutions in bipolar coordinates /13/ is considered, when, instead of wedge-shaped auxiliary domains, the **outsides** of circular crescents with vertices at the singular points /14/ are chosen. The functions obtained have the necessary nature of the singularities and, as distinct from the case of a wedge, satisfy the homogeneous equations in the initial multiply connected domain.

1. The method of compensating loads (MCL). In the domain $G \subset R^2$ with boundary Γ, we consider the boundary value problem

$$Lw(x) = f(x), \quad x \in G \tag{1.1}$$

$$\mathbf{l}w|_\Gamma = \boldsymbol{\varphi} \tag{1.2}$$

where L is a linear elliptic operator of order $2m$, for which we know the fundamental solution $K(x, y)$, $\mathbf{l} = (l_1, \ldots, l_m)$ are differential operators of the boundary conditions, and $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_m)$ is a vector function, given on Γ. We assume that the existence, uniqueness, and stability conditions hold for the solution of problem (1.1), (1.2). The domain G may be multiply connected and unbounded, while the operators $\mathbf{l}$ of the boundary conditions may be of mixed type.

The MCL consists of the following /1/. We consider a wider domain G' $(G \subset G')$ with a sufficiently smooth boundary Γ', located at a distance

$$d(\Gamma, \Gamma') = d > 0 \tag{1.3}$$

from Γ. Along the contour Γ' we apply distributed compensating loads with unknown densities $z_1(y), \ldots, z_m(y)$ (the specific form of the load will be indicated for each type of problem). The approximate solution of the problem is sought as

$$w_z(x) = \int_G K(x, y) f(y)\, dy + \sum_{j=1}^{m} \int_{\Gamma'} t_j K(x, y) z_j(y)\, ds_y \tag{1.4}$$

where the kernel $t_j K(x, y)$ is the solution of Eq.(1.1) for the corresponding unit loads concentrated at the point $y \in \Gamma'$. The kernels are obtained by applying the known differential operators t_j to the fundamental solution $K(x, y)$. The function $w_z(x)$ satisfies Eq.(1.1) exactly. On substituting it into boundary conditions (1.2) and using the spacing (1.3) between the contours Γ and Γ', we obtain the following system of integral equations of the 1st kind in the vector function $\mathbf{z}(y)$:

$$\sum_{j=1}^{m} \int_{\Gamma'} K_{ij}(x, y) z_j(y)\, ds_y = \varphi_i(x) - \int_G l_i K(x, y) f(y)\, dy \tag{1.5}$$

with continuous kernels $K_{ij}(x, y) = l_i t_j K(x, y)$, $i, j = 1, \ldots, m$.

The MCL is used in just the same way to solve elliptic systems (e.g., in the plane problem of the theory of elasticity). A feature of Eqs.(1.5) is that they may not have an exact solution $\mathbf{z}(y)$ /3-5/. For, if Γ and $f(x)$ are sufficiently smooth, the left-hand side of (1.5) is differentiable on Γ for any $\mathbf{z}(y) \in L_1(\Gamma')$, whereas $\boldsymbol{\varphi}(x)$ can only be a continuous vector function. It is all the more obvious that (1.5) will not have an exact solution if the required function $w(x)$ or its derivatives have singularities at corner points of the boundary Γ. We can therefore only speak of the approximate solution of system (1.5) in the sense of minimizing its discrepancy. It was shown in /3/ that the integral equations can have an indefinitely exact solution in the case of the three-dimensional problem of the theory of elasticity and Laplace's equation in a simply connected domain with a sufficiently smooth boundary.

It will be assumed below that, given any number $\varepsilon > 0$, an auxiliary contour Γ_ε' and a vector function $\mathbf{z}_\varepsilon \in L_2(\Gamma_\varepsilon')$ exist such that the norm of the discrepancy of (1.5) is less than ε /5/. For this, the solution $w(x)$ must have an ε-continuation from the domain G into G' /3/, i.e., given any $\varepsilon > 0$, there must be a solution $w_\varepsilon(x)$ of Eq.(1.1) in the domain G', for which we have the inequality $\|\mathbf{l}w_\varepsilon - \boldsymbol{\varphi}\|_\Gamma \leqslant \varepsilon$ in the corresponding norm on Γ.

Since the solution of integral equations of the 1st kind is an ill-posed problem, we have to use regularizing algorithms when realizing the MCL numerically. These topics were studied in /3-6/ as applied to MCL, so that they will not be discussed here. Even when regularization methods are used, however, we cannot use MCL effectively directly for problems with singularities on the boundary /9-11/. It is best to use the method of isolation of the singularities /7-12/.

Let the boundary Γ of the domain G contain a corner point Q with angle $2\beta_0$ between the tangents to Γ. We know /12/ that, in the neighbourhood of the point Q, the solution of problem (1.1), (1.2) (when $f(x)$ and (x) are infinitely differentiable and Γ is close to Q) can be written as the sum of an asymptotic series and an infinitely differentiable function $v_0(x)$

$$w(x) = v_0(x) + \sum_{k=1}^{\infty} v_k(x), \quad x \in G \cap U(Q) \tag{1.6}$$

The functions $v_k(x)$ are the solutions of homogeneous boundary value problems for certain model domains (e.g., infinite wedge-shaped domains /7, 8/) and depend only on the angle $2\beta_0$ and the type of boundary conditions. For the problems of the theory of elasticity considered, methods of constructing these solutions are well-known (see /7-12/ and the references quoted there). In representation (1.6) only those solutions which lead to finite energy /12/ appear.

The approximate solution of the boundary value problem is now sought as

$$w_{c,z}(x) = \sum_{k=1}^{N_Q} c_k v_k(x) + w_z(x) \tag{1.7}$$

where $\mathbf{c} = (c_1, \ldots, c_{N_Q})$ are unknown coefficients, $w_z(x)$ has the form (1.4), and the number N_Q is chosen in such a way that, in the solution remaining in (1.6) after subtracting N_Q terms of the series, there is no singularity in the DDS components /10/. The integrofunctional equations which appear, when $w_{c,z}(x)$ is substituted into boundary conditions (1.2), are solved by the method of regularization with respect to the coefficients $\mathbf{c}$ and densities $\mathbf{z}$ /9, 10/. Then, the DDS components inside G and on Γ are found by direct application (by virtue of condition (1.3)) of suitable differential operators to the approximate solution $w_{c,z}(x)$.

The method is considered separately below for each type of boundary value problem.

2. Harmonic problems. Let the domain G be bounded by $p+1$ closed contours $\Gamma = \Gamma_1 \cup \ldots \cup \Gamma_{p+1}$, $p \geqslant 0$, where Γ_{p+1} is the outer contour which embraces the rest with $p > 0$. Correspondingly, Γ' consists of contours $\Gamma_1', \ldots, \Gamma_{p+1}'$. The approximate solution of Poisson's equation $-\Delta w(x) = f(x)$, $x \in G$ is sought as the sum of the particular solution

$$w_0(x) = \int_G K(x,y) f(y)\, dy = \int_G \frac{1}{2\pi} \ln \frac{1}{|x-y|} f(y)\, dy$$

and a simple layer potential /3, 4/ (t_1 is the identity operator on Γ') or a double layer potential /9/ ($t_1 = \partial/\partial \mathbf{n}_y$ is the operator of differentiation with respect to the normal to Γ' at the point y).

If the corner point Q is on Γ_{p+1} (Fig.1), we consider an infinite wedge-shaped domain, bounded by the tangents to Γ_{p+1} at the point Q and having an angle $2\beta_0$. The singular solutions of Laplace's equation, bounded in the neighbourhood of the corner point, are sought in the form

$$v_k(r, \theta) = r^{\lambda_k} [A_k \cos \lambda_k \theta + B_k \sin \lambda_k \theta], \quad \lambda_k > 0 \tag{2.1}$$

where r, θ are polar coordinates with origin at the point Q and the bisector of the angle as polar axis /12/. The numbers λ_k, A_k, B_k are found by substituting (2.1) into the homogeneous boundary conditions for $\theta = \pm\beta_0$. For the Dirichlet and Neumann problems $\lambda_k = \pi k/(2\beta_0)$, $k = 1, \ldots$, and for the mixed problem ($\partial w(r, \beta_0)/\partial\theta = 0$, $w(r, -\beta_0) = 0$) $\lambda_k = -\pi/(4\beta_0) + \pi k/(2\beta_0)$, $k = 1, \ldots$. The coefficients A_k and B_k are given in /12/.

In (1.7) it is sufficient to take just the first term of expansion (2.1) with $0 < \lambda_1 < 1$ when $2\beta_0 > \pi$ (the Dirichlet and Neumann problems) or when $2\beta_0 > \pi/2$ (the mixed problem). In this case the derivatives in the singular solution $v_1(x)$ are not bounded at the point Q.

If, however, the corner point Q is on an interior component Γ_i, $1 \leqslant i \leqslant p$, of the boundary, the singular solutions of type (2.1) prove to be discontinuous in the domain G, since, on circuiting the point Q along a closed contour lying in G or along Γ_{p+1}, the functions (2.1) return at the start of the circuit to a new value. For this case, we propose the following /14/. We consider a symmetric circular crescent of size $2a$ with one vertex at the point Q and the other at Q_1 inside the contour Γ_i ($Q_1 \notin \bar{G}$) and the angle $2\beta_0$ between the tangents (Fig.2). Noting that the nature of the singularities of the singular solutions v_k depends only on the angle of opening and the boundary conditions on Γ_i close to the point Q, we choose the exterior of this crescent instead of the wedge-shaped domain. The functions v_k are written in modified bipolar coordinates /12/

$$x_1 = h \sin\beta, \quad x_2 = h \operatorname{sh}\alpha, \quad h = a/(\operatorname{ch}\alpha - \cos\beta) \tag{2.2}$$
$$x = (x_1, x_2), \quad -\infty < \alpha < +\infty, \quad -\beta_0 \leqslant \beta \leqslant \beta_0$$

The solutions $v_k(\alpha, \beta)$ are sought as products $\rho_k(\alpha)\omega_k(\beta)$. On applying to $v_k(\alpha, \beta)$ the

Laplace operator $\Delta = h^{-2}(\partial^2/\partial\alpha^2 + \partial^2/\partial\beta^2)$ and noting that the solution must be bounded in the neighbourhood of the point Q (as $\alpha \to +\infty$), we have

$$v_k(\alpha, \beta) = \exp(-\lambda_k\alpha)[A_k \cos\lambda_k\beta + B_k \sin\lambda_k\beta] \tag{2.3}$$

On substituting (2.3) into the boundary conditions with $\beta = \pm\beta_0$, we obtain expressions for λ_k, A_k, B_k, similar to those considered. Obviously, the functions $v_k(\alpha, \beta)$ and their derivatives are continuous in the exterior of the crescent. They are not bounded as we approach the vertex Q_1 (as $\alpha \to -\infty$), but Q_1 is located in the complement to $\bar{G}$, so that the $v_k(\alpha, \beta)$ are bounded in $\bar{G}$.

3. The plane problem of the theory of elasticity. Let the domain G satisfy the conditions of Sect.2. Plane deformation is described by an elliptic system of differential equations in the displacements $\mathbf{u}(x) = (u_1(x), u_2(x))$ /15/

$$(\lambda + \mu)\partial\Theta/\partial x_n + \mu\Delta u_n + X_n = 0, \quad n = 1, 2, \tag{3.1}$$

$$\Theta = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}, \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}$$

(E is the modulus of elasticity, ν is Poisson's ratio, and X_1 and X_2 are forces per unit volume). On Γ we pose the two boundary conditions.

As the compensating loads we take forces $z_1(y)$, $z_2(y)$, distributed over Γ' and lying in the $x_1 0 x_2$ plane. The fundamental matrix $K_{ij}(x, y)$ $(i, j = 1, 2)$ is found from the well-known expressions for the stresses and displacements under the action of concentrated forces on the plane /15/. The relations for the plane stressed state are obtained in the usual way by replacing λ by $\lambda^* = E\nu/(1-\nu)$.

To improve the stability of the numerical realization of MCL, we recommend that the auxiliary contour Γ' be a step-line, and the compensating loads z_1, z_2 be piecewise polynomial. Then, for the basic types of boundary value problems of the plane theory of elasticity, the integrals on the left-hand side of Eq.(1.5) can be evaluated in closed form.

Methods of isolating the singularities for infinite wedge-shaped domains (when $Q \in \Gamma_{p+1}$) have been studied in /7, 8, 11, 12/ etc. For instance, Airy's function in polar coordinates r, θ has the form /7, 8/

$$U_k(r, \theta) = r^{\lambda_k+1}[A_k \cos(\lambda_k + 1)\theta + B_k \sin(\lambda_k + 1)\theta + C_k \cos(\lambda_k - 1)\theta + D_k \sin(\lambda_k - 1)\theta] = r^{\lambda_k+1} g_k(\theta) \tag{3.2}$$

where the numbers λ_k (in general complex) are found by solving the appropriate transcendental equations. We have to leave in (1.7) the singular solutions v_k with numbers $0 < \operatorname{Re}\lambda_k < 1$, to which there correspond, for critical values of the angles $2\beta_0$ stresses which are unbounded at the point Q, while the condition for finite energy is satisfied /12/.

To find the singular solutions in bipolar coordinates (2.2) we use the following forms of the biharmonic operator /13/:

$$\Delta^2 U(\alpha, \beta) = \frac{1}{h^3}\left[\frac{\partial^4}{\partial\alpha^4} + 2\frac{\partial^4}{\partial\alpha^2\partial\beta^2} + \frac{\partial^4}{\partial\beta^4} - 2\frac{\partial^2}{\partial\alpha^2} + 2\frac{\partial^2}{\partial\beta^2} + 1\right]\left(\frac{U}{h}\right)$$

Using the method of separation of variables, similar to that described in Sect.2, for the function U/h, we find the stress function

$$U_k(\alpha, \beta) = h \exp(-\lambda_k\alpha) g_k(\beta) \tag{3.3}$$

where the $g_k(\beta)$ have the form (3.2). While the nature of the singularities of the functions $U_k(\alpha, \beta)$ at the point Q (as $\alpha \to +\infty$) is the same as in (3.2), the $U_k(\alpha, \beta)$ are single-valued in the exterior of the crescent. The expressions for the components σ_α^k, σ_β^k, $\tau_{\alpha\beta}^k$ of the stress tensor are obtained from the Airy function U_k be the relations /13/

$$a\sigma_\alpha = \left[\frac{a}{h}\frac{\partial^2}{\partial\beta^2} - \operatorname{sh}\alpha\frac{\partial}{\partial\alpha} - \sin\beta\frac{\partial}{\partial\beta} + \operatorname{ch}\alpha\right]\left(\frac{U}{h}\right)$$

$$a\sigma_\beta = \left[\frac{a}{h}\frac{\partial^2}{\partial\alpha^2} - \operatorname{sh}\alpha\frac{\partial}{\partial\alpha} - \sin\beta\frac{\partial}{\partial\beta} + \cos\beta\right]\left(\frac{U}{h}\right)$$

$$a\tau_{\alpha\beta} = -\frac{a}{h}\frac{\partial^2}{\partial\alpha\partial\beta}\left(\frac{U}{h}\right)$$

To evaluate the displacements, we use their complex forms /15/. We put $z = x_1 + ix_2$, $\gamma = \alpha + i\beta$. Relations (2.2) are rewritten as $z = ai \operatorname{cth} {}^1/_2\gamma$. In the notation of /15/ we have $P_k(\gamma) = \Delta U_k(\gamma)$, $f_k(\gamma) = P_k(\gamma) + iQ_k(\gamma)$, where $Q_k(\gamma)$ is the harmonic function conjugate to $P_k(\gamma)$. It can be found, apart from a constant, from /15/

$$Q_k(\gamma) = -\int_{\alpha_0}^{\alpha}\frac{\partial P_k}{\partial\beta}(\alpha, \beta_0)\, d\alpha + \int_{\beta_0}^{\beta}\frac{\partial P_k}{\partial\alpha}(\alpha, \beta)\, d\beta$$

We finally obtain $f_k = 4\mu\,(\Lambda_k^+\xi_k - \Lambda_k^-\zeta_k)$, where $\xi_k = A_k + iB_k$, $\zeta_k = C_k + iD_k$, $\Lambda_k^\pm = (\lambda_k \pm 1)\exp(-\lambda_k\gamma) - \lambda_k \exp[-(\lambda_k \pm 1)\gamma]$. The functions $\varphi_k(\gamma)$ (see /15/) have the form

$$\varphi_k(\gamma) = \frac{1}{4}\int f_k(z)\,dz = -\frac{1}{4}\int f_k(\gamma)\,\frac{ai\,d\gamma}{\operatorname{ch}\gamma - 1} =$$
$$\frac{2\mu ai}{\exp\gamma - 1}\{\exp(-\lambda_k\gamma)\xi_k + \exp[-(\lambda_k - 1)\gamma]\zeta_k\}$$

Notice that we can write the stress function $U_k(\gamma)$ as

$$U_k(\gamma) = \operatorname{Re}\Big\{\frac{4\mu a^2}{(\exp\gamma - 1)(\overline{\exp\gamma} - 1)} \times$$
$$[\overline{\exp\gamma}\exp(-\lambda_k\gamma)\xi_k + \exp[-(\lambda_k - 1)\gamma]\zeta_k]\Big\}$$

(the bar denotes the complex conjugate). From this and the relation $U_k = \operatorname{Re}[\bar{z}\varphi_k(z) + \chi_k(z)]$ we have

$$\chi_k(\gamma) = 2\mu a^2(\exp\gamma - 1)^{-1}\{\exp(-\lambda_k\gamma)\xi_k - \exp[-(\lambda_k - 1)\gamma]\zeta_k\}$$

On differentiating with respect to z, we have for $\psi_k(z) = \chi_k'(z)$ $\psi_k(\gamma) = -\mu ai\,(\Lambda_k^+\xi_k + \Lambda_k^-\zeta_k)$.

The expressions for the displacements u_1, u_2, and u_α, u_β, are obtained from the relations /15/

$$2\mu\,(u_1 + iu_2) = \varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)},\quad \varkappa = 3 - 4\nu \tag{3.4}$$

and the easily obtained expressions $h\,(\overline{\operatorname{ch}\gamma} - 1)(u_\alpha + iu_\beta) = ia\,(u_1 + iu_2)$.

The components of the displacements u_1, u_2 in (3.4) are given in the x_1, x_2 coordinate system, the origin of which is at the mid-point of the segment QQ_1, while the $0x_2$ axis passes through the crescent vertices Q and Q_1 (Fig.2). All the DDS components found above can be written in this coordinate system by using the replacement $\exp\gamma = (z + ia)(z - ia)^{-1}$. From the relations obtained there follows the uniqueness of the DDS components of the singular solutions in the exterior of the crescent, and hence, in the domain G.

4. The problem of the bending of a plate. The definition $w(x)$ of a thin elastic plate satisfies the Sophie Germain equation

$$\Delta^2 w(x) = q(x)/D,\quad x \in G \tag{4.1}$$

where D is the cylindrical rigidity of the plate, and $q(x)$ is the density of the distributed normal load. On the boundary Γ we pose two boundary conditions. As compensating loads we apply to Γ' normal forces $z_1(y)$ and bending moments $z_2(y)$ /1, 5, 6/. The corresponding fundamental solutions are /5, 6/

$$K_1(x, y) = (8\pi D)^{-1}\,|x - y|\ln|x - y|^2,\quad K_2(x, y) = -\partial K_1(x, y)/\partial n_y$$

The expressions for the singular solutions of the homogeneous biharmonic equation are in fact written in Sect.3 ((3.2), (3.3)) /10, 14/. The other components of the DDS are found in polar and bipolar coordinates by using the well-known differential operators. In (1.7) we include the singular solutions with numbers $0 < \operatorname{Re}\lambda_k < 2$. These solutions have singularities in the moments (with $0 < \operatorname{Re}\lambda_k < 1$) and in the shearing forces.

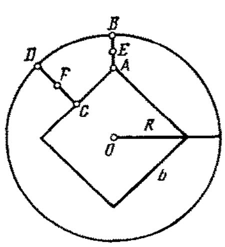

Fig.3

Table 1

N_Q	$c_1 \cdot 10^3$	$c_2 \cdot 10^3$	$c_3 \cdot 10^4$	δ_1, %	δ_2, %
0				28.8	26.2
1	3.80			1.8	0.56
2	3.37	2.26		0.20	0.08
3	3.35	2.45	0,09	0.11	0.08

As an example of our method, we will solve the problem of the bending of a plate (Fig.3) fixed along its contour, under the action of a uniformly distributed load with density $q(x)/D \equiv 1$. The radius of the circle $R = 1$, and the length of the side of the square is $b = 0{,}5$. The particular solution of Eq.(4.1) is $w_0(x) = (64D)^{-1}(x_1^2 + x_2^2)^2$. In the numerical solution we took piecewise constant compensating loads and the integrals were evaluated in closed form. We used the symmetry of the problem to lower the order of the system of linear algebraic equations obtained.

To isolate the singularities, we inscribed in each of the four reentrant angles the crescent $(2\beta_0 = 3\pi/2, 2a = 0.2)$. The numbers λ_k are found by solving the equation /8, 11/ $\lambda_k \sin 2\beta_0 \pm \sin(2\beta_0\lambda_k)$. By the symmetry with respect to the bissector of the reentrant angles, we included in (1.7) only the symmetric solutions (corresponding to the plus sign in the equation) with numbers $0 < \operatorname{Re}\lambda_k < 2$: $\lambda_1 = 0.544484$, $\lambda_2 = 1.62926 + i0{,}231251$ $v_k(\alpha, \beta) = h\exp(-\lambda_k\alpha)[\cos(\lambda_k - 1)\beta_0\cos(\lambda_k + 1)\beta - \cos(\lambda_k + 1)\beta_0\cos(\lambda_k - 1)\beta]$.

For complex λ_2, we used $\operatorname{Re} v_2(\alpha, \beta)$ and $\operatorname{Im} v_2(\alpha, \beta)$. The distance Δ' to the auxiliary contour Γ' was varied in the range 0.01 - 0.3; the discrepancies of system (1.5) were least with $\Delta' = 0.1$. The present method was compared with the MCL without isolation of the singularities.

The computed results for $\Delta' = 0{,}1$ and different numbers N_Q of the singular solutions used are shown in Table 1 (the number of sections into which 1/8 of the symmetric part of Γ' was divided was 5, and the number of division points of the corresponding pieces of Γ was 20, see /5/). We denote by c_1, c_2, c_3 the coefficients obtained for the singular terms $v_k(\alpha, \beta)$; δ_1 and δ_2 are the ratios of the errors in the boundary conditions $w|_\Gamma = 0$, $\theta_n = \partial w/\partial n|_\Gamma = 0$ to the maximum w and θ inside the domain (in %). The maximum deflections in the segments AB and CD were $w_E = 5.753\cdot 10^{-4}$ ($|AE| = 0{,}45|AB|$), $w_F = 6{,}997\cdot 10^{-4}$ ($|CF| = 0.5|CD|$).

It follows from the results that isolation of the singularities improves the quality of the solution of problems in irregular domains when the MCL is used.

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